סמסטר ב', מועד א', תשס"ט
תאריך הבחינה: 08.07.2009
מספר קורס: 0366-3022

בחינה במבוא לאנליזה פונקציונלית

המורה: פרופ' בוריס צירלסון

משך הבחינה: 3 שעות.
רצוי לענות על כל השאלות.

בהצלחה!

שאלה 1

(א) מצאו את כל הקטעים $[a,b] \subset \mathbb{R}$ שעבורם פונקצית האינדיקטור

$$f = \mathbb{1}_{[a,b]} \in L_2(\mathbb{R})$$

מקיימת

$$\mathcal{F}(\mathcal{F}(f)) = f\,.$$

....

(ב) אותו הדבר עבור

$$\mathcal{F}(\mathcal{F}(f)) = -f\,.$$

רמז: הזכרו בהתמרת פוריה ההפוכה.

שאלה 2

תהיו $\varphi, \psi : \mathbb{R} \to \mathbb{C}$ פונקציות מדידות מקיימות $\varphi(x)\psi(x) = 1$ כמעט לכל $x \in \mathbb{R}$.
(א). נסחו והוכיחו יחס בין האופרטורים $\varphi(Q)$, $\psi(Q)$.

....

(ב). אותו הדבר עבור $\varphi(P)$, $\psi(P)$.

KKK-9

שאלה 3

יהיו $\lambda \in \mathbb{C}$, $|\lambda| = 1$, ו-U אופרטור אוניטרי. הוכחו או הפרכו:
אם $\mathbb{1}_{\{\lambda\}}(U) \neq 0$ אז קיים $x \neq 0$ כך ש-$Ux = \lambda x$.

שאלה 4

נגדיר דיסטריבוציה $T \in D_1'(-1,1)$ ע"י

$$\langle T, \varphi \rangle = \int_0^1 \frac{\varphi(x) - \varphi(0)}{x} \, \mathrm{d}x$$

לכל $\varphi \in D_1(-1,1)$. הוכחו או הפרכו:
(א) קיימת $f \in L_1(-1,1)$ כך ש-$f' = T$;

. . . .

(ב) קיימת $f \in L_\infty(-1,1)$ כך ש-$f' = T$.

SUMMARY: RESULTS AND FORMULAS

1k List of results

1k1 Plancherel's theorem

There exists a bounded linear operator (called Fourier transform) $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ such that

$$\mathcal{F}f : t \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}st} f(s) \, \mathrm{d}s \quad \text{for all } f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \, .$$

The operator $\mathcal{F}$ is isometric, which means the following.

Plancherel's formula:

$$\|\mathcal{F}f\| = \|f\| \quad \text{for all } f \in L_2(\mathbb{R}) \, .$$

Parseval's formula:

$$\langle \mathcal{F}f, \mathcal{F}g \rangle = \langle f, g \rangle \quad \text{for all } f, g \in L_2(\mathbb{R}) \, .$$

1k2 Inversion formula

The isometric operator $\mathcal{F}$ is unitary, which means that $\mathcal{F}\big(L_2(\mathbb{R})\big) = L_2(\mathbb{R})$. The inverse operator $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1} = \mathcal{F}J = J\mathcal{F}$$

where J is defined by $Jf : t \mapsto f(-t)$.

1k3 Diagonalization of shifts

$$\mathcal{F}U(a)\mathcal{F}^{-1} = V(a) \quad \text{for all } a \in \mathbb{R} \, ,$$

where operators $U(a), V(a) : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ are defined by

$$U(a)f : t \mapsto f(t + a) \, , \quad V(a)f : t \mapsto \mathrm{e}^{\mathrm{i}at} f(t) \, .$$

1k4 Diagonalization of the Fourier transform

There exists an orthonormal basis $(e_0, e_1, e_2, \dots)$ of $L_2(\mathbb{R})$ such that

$$\mathcal{F}e_k = (-\mathrm{i})^k e_k \quad \text{for } k = 0, 1, 2, \dots$$

Namely,

$$e_k = \frac{2^{k/2}}{\sqrt{k!}} \frac{\mathrm{d}^k}{\mathrm{d}z^k}\Big|_{z=0} \psi(z)\,,$$

where $\psi : \mathbb{C} \to L_2(\mathbb{R})$ is defined by

$$\psi(z) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + zq - \frac{z^2}{4} \Big)\,.$$

1k5 Some complete systems in $L_2(\mathbb{R})$

The vectors e_k mentioned above are a complete system in the sense that their linear combinations are dense in $L_2(\mathbb{R})$. Also functions

$$q \mapsto \exp\big(-(q-x)^2\big)$$

for all $x \in \mathbb{R}$ are a complete system.

1k6 Fourier transform and convolution

$$\mathcal{F}(f * g) = (2\pi)^{1/2} (\mathcal{F}f) \cdot (\mathcal{F}g) \quad \text{for all } f \in L_2(\mathbb{R}),\ g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,;$$

here

$$f * g : t \mapsto \int f(t-s) g(s)\,\mathrm{d}s\,.$$

1k7 Operators commuting with shifts

A bounded linear operator on $L_2(\mathbb{R})$ commutes with all $U(a)$ if and only if it is of the form

$$f \mapsto \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f)$$

for some $\varphi \in L_\infty(\mathbb{R})$.

1l List of formulas

$x, y \in \mathbb{R}$; $z_1, z_2 \in \mathbb{C}$; $\psi_0, w_0 : \mathbb{C} \to H$;

(1l1) $$\langle \psi_0(z_1), \psi_0(z_2) \rangle = \exp\Big(\frac{1}{2} z_1 \overline{z}_2 \Big)\,;$$

(1l2) $$w_0(z) = \frac{\psi_0(z)}{\|\psi_0(z)\|} = \mathrm{e}^{-|z|^2/4} \psi_0(z)\,;$$

(1l3) $$\langle w_0(z_1), w_0(z_2) \rangle = \exp\Big(-\frac{1}{4}|z_1 - z_2|^2 + \frac{\mathrm{i}}{2} \mathrm{Im}\,(z_1 \overline{z}_2) \Big)\,;$$

(1l4) $$\langle w_0(x), w_0(y) \rangle = \mathrm{e}^{-|x-y|^2/4} = \langle w_0(\mathrm{i}x), w_0(\mathrm{i}y) \rangle\,;$$

$U_0, V_0 : \mathbb{R} \to \mathrm{Unitary}(H)$;

(1l5) $$U_0(a)w_0(x) = w_0(x-a)\,;$$

(1l6) $$V_0(b)w_0(\mathrm{i}y) = w_0\big(\mathrm{i}(y+b)\big)\,;$$

$\psi_1, w_1 : \mathbb{C} \to L_2(\mathbb{R})$;

(1l7) $$\psi_1(z) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + zq - \frac{z^2}{4}\Big)\,;$$

(1l8) $$w_1(z) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + zq - \frac{z^2}{4} - \frac{|z|^2}{4}\Big)\,;$$

(1l9) $$w_1(x) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{1}{2}(q-x)^2\Big)\,;$$

(1l10) $$w_1(\mathrm{i}y) : q \mapsto \pi^{-1/4} \exp\Big(-\frac{q^2}{2} + \mathrm{i}yq\Big)\,;$$

$U_1, V_1 : \mathbb{R} \to \mathrm{Unitary}(L_2(\mathbb{R}))$, $a, b \in \mathbb{R}$;

(1l11) $$U_1(a)f : q \mapsto f(q+a)\,;$$

(1l12) $$V_1(b)f : q \mapsto \mathrm{e}^{\mathrm{i}bq} f(q)\,;$$

$\mathcal{F} \in \mathrm{Unitary}(L_2(\mathbb{R}))$;

(1l13) $$\mathcal{F}\psi_1(z) = \psi_1(-\mathrm{i}z)\,;$$

(1l14) $$\mathcal{F}U_1(a)\mathcal{F}^{-1} = V_1(a)\,;$$

(1l15) $$\mathcal{F}^{-1} = \mathcal{F}J = J\mathcal{F}\,, \quad Jf : q \mapsto f(-q)\,;$$

(1l16) $$\mathcal{F}f : t \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}st} f(s)\,\mathrm{d}s \quad \text{for } f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,;$$

(1l17) $$f * g = \int g(a) U_1(-a) f \,\mathrm{d}a : t \mapsto \int f(t-s) g(s)\,\mathrm{d}s\,;$$

(1l18) $$f * g = g * f \quad \text{for } f, g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,;$$

(1l19) $$\mathcal{F}(f * g) = (2\pi)^{1/2} (\mathcal{F}f) \cdot (\mathcal{F}g) \quad \text{for } f \in L_2(\mathbb{R}),\ g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,.$$

2e List of formulas

Multiplication operators:

(2e1) $$Qf : q \mapsto qf(q) \quad \text{for } f \in D_Q\,;$$

(2e2) $$\varphi(Q)f = \varphi \cdot f : q \mapsto \varphi(q)f(q) \quad \text{for } f \in D_{\varphi(Q)}\,;$$

(2e3) $$\exp(\mathrm{i}bQ) = V(b)\,;$$

(2e4) $$Qf = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}b}\bigg|_{b=0} V(b)f \quad \text{for } f \in D_Q\,;$$

(2e5) $$E^{(Q)}_{a,b} = \mathbb{1}_{(a,b)}(Q)\,;$$

(2e6) $$\|E^{(Q)}_{a,b}f\|^2 = \langle E^{(Q)}_{a,b}f, f\rangle = \int_a^b |f(q)|^2\,\mathrm{d}q\,.$$

Operators commuting with shifts:

(2e7) $$P = \mathcal{F}^{-1}Q\mathcal{F}\,;$$

(2e8) $$Pf : q \mapsto -\mathrm{i}f'(q) \quad \text{for nice } f\,;$$

(2e9) $$\varphi(P) = \mathcal{F}^{-1}\varphi(Q)\mathcal{F} : f \mapsto \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f)\,;$$

(2e10) $$\exp(\mathrm{i}aP) = U(a)\,;$$

(2e11) $$Pf = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}a}\bigg|_{a=0} U(a)f \quad \text{for } f \in D_P\,;$$

(2e12) $$E^{(P)}_{a,b} = \mathbb{1}_{(a,b)}(P) = \mathcal{F}^{-1}E^{(Q)}_{a,b}\mathcal{F}\,;$$

(2e13) $$\|E^{(P)}_{a,b}f\|^2 = \langle E^{(P)}_{a,b}f, f\rangle = \int_a^b |(\mathcal{F}f)(p)|^2\,\mathrm{d}p\,;$$

(2e14) $$E^{(P)}_{a,b}f = \Big(q \mapsto \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}bq} - \mathrm{e}^{\mathrm{i}aq}}{q}\Big) * f \quad \text{for } f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})\,.$$

3j List of results

3j1 Theorem. For every self-adjoint operator A there exists a unique unitary operator U (so-called Cayley transformed of A) such that

(a) $U(A + \mathrm{i}\mathbb{1})x = (A - \mathrm{i}\mathbb{1})x$ for all $x \in \mathrm{Domain}(A)$;

(b) $\mathrm{Range}(\mathbb{1} - U) = \mathrm{Domain}(A)$, and $A(\mathbb{1} - U)x = \mathrm{i}(\mathbb{1} + U)x$ for all $x \in H$.

3j2 Theorem. For every unitary operator U there exists a unique positive $*$-homomorphism $f \mapsto f(U)$ from $C(\mathbb{T})$ to bounded operators, such that $\|f(U)\| \le \|f\|$ for all f, and if $\forall z \in \mathbb{T}\;\; f(z) = z$ then $f(U) = U$.

3j3 Theorem. For every self-adjoint operator A there exists a unique positive $*$-homomorphism $f \mapsto f(A)$ from $C(\mathbb{R} \cup \{\infty\})$ to bounded operators, such that $\|f(A)\| \le \|f\|$ for all f, and if $\forall a \in \mathbb{R}\ f(a) = \frac{a-\mathrm{i}}{a+\mathrm{i}}$ then $f(A)$ is the Cayley transformed of A.

3j4 Theorem. The $*$-homomorphism of 3j2 has a unique extension from $C(\mathbb{T})$ to the $*$-algebra $L(\mathbb{T})$ (spanned by all bounded semicontinuous functions) satisfying

$\|f(U)\| \le \sup |f(\cdot)|$ for all $f \in L(\mathbb{T})$;

$C(\mathbb{T} \to \mathbb{R}) \ni f_n \uparrow f \in L(\mathbb{T} \to \mathbb{R})$ implies $\forall x \in H\ \ f_n(U)x \to f(U)x$.

3j5 Theorem. The $*$-homomorphism of 3j3 has a unique extension from $C(\mathbb{R} \cup \{\infty\})$ to the $*$-algebra $L(\mathbb{R})$ (spanned by all bounded semicontinuous functions) satisfying

$\|f(A)\| \le \sup |f(\cdot)|$ for all $f \in L(\mathbb{R})$;

$C(\mathbb{R} \cup \{\infty\} \to \mathbb{R}) \ni f_n \uparrow f \in L(\mathbb{R} \to \mathbb{R})$ implies $\forall x \in H\ \ f_n(A)x \to f(A)x$;

if $\forall n, a\ \ f_n(a) = a\mathbb{1}_{(-n,n)}(a)$ then $\forall x \in \mathrm{Domain}(A)\ \ f_n(A)x \to Ax$.

3j6 Theorem. The formulas

$$U_t = \exp(\mathrm{i}tA)\,,$$
$$Ax = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} U_t x$$

establish a one-to-one correspondence between all strongly continuous one-parameter unitary groups (U_t) and all self-adjoint operators A.

3j7 Theorem. For every continuous function $v : \mathbb{R} \to \mathbb{R}$ bounded from below there exists one and only one strongly continuous one-parameter unitary group (U_t) on $L_2(\mathbb{R})$ such that

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U_t \psi = -\psi'' + v \cdot \psi$$

for all twice continuously differentiable, compactly supported functions $\psi : \mathbb{R} \to \mathbb{C}$.

4e List of formulas

T is a distribution, φ is a test function.

(4e1) $$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle \,;$$

(4e2) $$\langle \delta_x, \varphi \rangle = \varphi(x) \,;$$

(4e3) $$\big(1\!\!1_{(a,b)}\big)' = \delta_a - \delta_b \,;$$

(4e4) $$\big(\delta_x^{(m)}\big)' = \delta_x^{(m+1)} \,;$$

(4e5) $$\langle \delta_x^{(m)}, \varphi \rangle = (-1)^m \varphi^{(m)}(x) \,;$$

(4e6) $$\delta_x(\alpha(\cdot)) = (\alpha^{-1})'(x) \delta_{\alpha^{-1}(x)} \,.$$

$\mathcal{F}$ is the Fourier transform, "$*$" means convolution.

(4e7) $$\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle \,;$$

(4e8) $$\mathcal{F}P = Q\mathcal{F} \,; \qquad P\mathcal{F} = -\mathcal{F}Q \,;$$

(4e9) $$\mathcal{F}\delta_0 = (2\pi)^{-1/2} \cdot 1\!\!1 \,; \qquad \mathcal{F}1\!\!1 = (2\pi)^{1/2} \delta_0 \,;$$

(4e10) $$\mathcal{F}\delta_x : y \mapsto (2\pi)^{-1/2} \mathrm{e}^{-\mathrm{i}xy} \,; \qquad \mathcal{F}\big(y \mapsto \mathrm{e}^{-\mathrm{i}xy}\big) = (2\pi)^{1/2} \delta_{-x} \,;$$

(4e11) $$\mathcal{F}\delta_0' : x \mapsto \mathrm{i}(2\pi)^{-1/2} x \,; \qquad \mathcal{F}(x \mapsto x) = \mathrm{i}(2\pi)^{1/2} \delta_0' \,;$$

(4e12) $$\mathcal{F}\delta_0^{(m)} : x \mapsto \mathrm{i}^m (2\pi)^{-1/2} x^m \,; \qquad \mathcal{F}\big(x \mapsto x^m\big) = \mathrm{i}^m (2\pi)^{1/2} \delta_0^{(m)} \,;$$

(4e13)
$$\mathcal{F}\delta_x^{(m)} : y \mapsto \mathrm{i}^m (2\pi)^{-1/2} y^m \mathrm{e}^{-\mathrm{i}xy} \,; \qquad \mathcal{F}\big(y \mapsto y^m \mathrm{e}^{-\mathrm{i}xy}\big) = \mathrm{i}^m (2\pi)^{1/2} \delta_{-x}^{(m)} \,;$$

(4e14) $$T * \delta_0' = T' \,;$$

(4e15) $$T * \delta_0^{(m)} = T^{(m)} \,;$$

(4e16) $$\delta_0^{(m)} * \delta_0^{(n)} = \delta_0^{(m+n)} \,.$$